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Spectral laws for the enstrophy cascade in a two-dimensional turbulence

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Abstract. We use a generalised Von Karman-Heisenberg-von Weizsacker-type model for the inertial transfer to give a generalised spectral law for the enstrophy cascade in a two-dimensional turbulence that exhibits a steeper energy spectrum for large wavenumbers and reduces to the well known k^{-3} spectrum at the other end of the spectrum. For very high wavenumbers, this spectrum is, in fact, an arbitrarily steep power law. Nonetheless, it is possible to given an even more rapidly decaying spectrum for this range, using a continuous spectral cascading model. We will then discuss the intermittency aspects of the departures from the Batchelor-Kraichnan scaling law and show that while the intermittency corrections within the framework of the β model of Frisch *et al* are in qualitative agreement with the predictions made by the generalised spectral law given in this paper, intermittency by itself is unable to account fully for the steeper spectra at large wavenumbers observed in the numerical experiments. We will discuss further fractal aspects of the enstrophy cascade, and show that for the enstrophy cascade, the fractal dimension rules not only the manner in which the cascading proceeds but also the point where it stops.

1. Introduction

Kraichnan (1967) and Batchelor (1969) pointed out the possibility of two inertial ranges in a two-dimensional turbulence: a $k^{-5/3}$ spectrum range in which the energy propagates to larger scales, and k^{-3} spectrum range in which the enstrophy (which is the square of vorticity) propagates to smaller scales. Kraichnan (1967) proposed that both ranges would exist simultaneously in a continuously driven turbulence. Leith (1968) derived a diffusion approximation to inertial energy transfer in such a way that energy and enstrophy are conserved, and predicted the $k^{-5/3}$ and k^{-3} inertial ranges. Lilly (1969), Herring *et al* (1974), Pouquet *et al* (1975), and Frisch and Sulem (1984) carried out numerical simulation experiments and confirmed the conjecture of Kraichnan (1967) and Batchelor (1969) that there occurs a transfer of excitation to lower and higher wavenumbers in a manner qualitatively consistent with the simultaneous existence of both the energy and enstrophy inertial ranges. Atmospheric measurements have also revealed the existence of an energy cascade (Fjortoft 1953) and an enstrophy cascade (Ogura 1958, Wiin-Nielsen 1967, Julian *et al* 1970, Morel and Necco 1973, Morel and Larcheveque 1974, Desbois 1975). Kraichnan (1971) proposed further that the k^{-3} spectrum for the enstrophy cascade should be modified by a logarithmic correction term to give $k^{-3}[\ln(k/k_c)]^{-1/3}$. However, the latter result does not extend to infinity, because it does not give the rapid decay of the spectrum prevalent at high wavenumbers. Kida (1981) applied the modified cumulant expansion and numerically calculated the equations for the energy spectrum, and confirmed a more rapid decay

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of the spectrum in the enstrophy cascade at very large wavenumbers. Numerical simulations of decaying flows (Basdevant and Sadourny 1983, McWilliams 1984a) and forced flows (Kida 1985, Brachet *et al* 1986, Basdevant *et al* 1981) also gave energy spectra steeper than k^{-3}.

The whole theory of two-dimensional turbulence had, until recently, remained almost an academic exercise, notwithstanding its possible connections with atmospheric and oceanic large-scale flows. Just recently, truly two-dimensional flows were produced to a close approximation finally in laboratory experiments. Experimental evidence of the existence of inverse energy cascade was first obtained by Couder (1984) on thin liquid soap films, then by Sommeria (1986) in a shallow mercury layer immersed in a strong normal magnetic field. However, the enstrophy cascade has not been obtained in the laboratory, and at present only numerical simulations have been able to give some information on it.

In this paper, we will use a generalised Von Karman–Heisenberg–von Weizsacker-type inertial transfer model to give a generalised spectral law for the enstrophy cascade that exhibits a steeper energy spectrum for large wavenumbers and reduces to the well known k^{-3} spectrum at the other end of the spectrum. For very high wavenumbers this spectrum is, in fact, an arbitrarily steep power law. Nonetheless, it is possible to give an even more rapidly decaying exponential-type spectrum, using a stationary continuous spectral cascading model.

Basdevant *et al* (1981) argued that the steeper energy spectra at large wavenumbers is due to intermittency in the flow: enstrophy dissipation is a highly fluctuating quantity whose statistical properties significantly affect the energy spectrum at small scales. Earlier, Mandelbrot (1976) argued that intermittency is related to the fractal aspects of turbulence. In particular, Mandelbrot (1976) proposed that the dissipation is concentrated on a set with non-integer Hausdorff dimension. Mandelbrot's ideas were formulated in a simpler way through a phenomenological model called the β-model (which was based on the ideas advanced by Kraichnan 1972) by Frisch *et al* (1978). The key assumption in this model is that the flux of energy is transferred to only a fixed fraction β of the eddies downstream in the cascade. A noteworthy feature of the β-model is that we do not have to assume the Batchelor–Kraichnan scaling laws initially and then derive their modified versions by somehow mysteriously incorporating the dissipation fluctuations.

The application of the β-model to the inverse energy cascade was done by Frisch *et al* (1978), who found that the intermittency corrections decrease the 5/3 exponent. In this paper, we will apply the β-model to the enstrophy cascade and confirm that intermittency will steepen the energy spectrum, in qualitative agreement with the generalised spectral law for the enstrophy cascade given in this paper. However, we will show that intermittency by itself is unable to account fully for steeper spectra observed in the numerical experiments. We will finally consider further fractal aspects of the enstrophy cascade and show that the fractal dimension now rules not only the manner in which the cascade proceeds but also the point where it stops.

2. Fourier analysis of the turbulent field

Fourier analysis of the velocity field, when it is a stationary random function of position, affords a view of the turbulent motion as comprised of the superposition of motions of a large number of components of different scales. These components contribute additively to the total energy and total enstrophy and interact with each other according to the nonlinear inertial terms in the equations of flow.

Let us express the flow properties at any point x at time t, as a superposition of plane waves of the form,

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{X}, t) &= \sum_{\boldsymbol{k}} \boldsymbol{V}(\boldsymbol{k}, t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ \frac{1}{\rho} p(\boldsymbol{x}, t) &= \sum_{\boldsymbol{k}} P(\boldsymbol{k}, t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \end{aligned} \tag{1}$$

where $\boldsymbol{v}$ is the fluid velocity, p the pressure, and ρ is the fluid density. Since $\boldsymbol{V}$ and P are actually measurable, they must be real so that

$$\boldsymbol{V}^*(\boldsymbol{k}) = \boldsymbol{V}(-\boldsymbol{k}) \qquad P^*(\boldsymbol{k}) = P(-\boldsymbol{k}). \tag{2}$$

We have dropped the argument t for convenience. We then obtain from the equations of continuity and motion

$$\nabla \cdot \boldsymbol{v} = 0 \tag{3}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{v} \tag{4}$$

and the equation

$$\left(\frac{\partial}{\partial t} + \nu|\boldsymbol{k}|^2\right) V_j(\boldsymbol{k}) = -\mathrm{i}k_m\left(\delta_{jl} - \frac{k_j k_l}{k^2}\right) \sum_{\boldsymbol{k}'} V_m(\boldsymbol{k}') V_l(\boldsymbol{k}-\boldsymbol{k}'). \tag{5}$$

Here ν is the kinematic viscosity.

In two-dimensional turbulence, there are two conserved quantities—the energy and the enstrophy which is the mean square vorticity. (Due to a finite viscosity, however, the enstrophy is dissipated at a non-negligible rate; therefore the maintenance of a stationary state requires an external source since the vortex stretching, which acts like a source of vorticity, is inoperative here unlike the three-dimensional turbulence. However, energy dissipation will tend to zero as $\nu \to 0$ so that two-dimensional turbulence is almost non-dissipative as $\nu \to 0$.) Therefore, there are two types of inertial ranges—one for energy and one for enstrophy.

3. Generalised Von Karman–Heisenberg–von Weizsacker-type inertial transfer model for the enstrophy cascade

We have from (5)

$$\left(\frac{\partial}{\partial t} + \nu|\boldsymbol{k}|^2\right) D(\boldsymbol{k}) = \sum_{\boldsymbol{k}} U_{ml}(\boldsymbol{k}, \boldsymbol{k}') \tag{6}$$

where $D(\boldsymbol{k})$ is the enstrophy density in the Fourier space,

$$D(\boldsymbol{k}) = \tfrac{1}{2}|\Omega(\boldsymbol{k})|^2 = \tfrac{1}{2}|\boldsymbol{k} \times \boldsymbol{V}(\boldsymbol{k})|^2 = k^2 \xi(\boldsymbol{k}) \tag{7}$$

and

$$U_{ml} = -\mathrm{i}k_m \Omega_l(\boldsymbol{k}) V_m(\boldsymbol{k}') \Omega_l(\boldsymbol{k}-\boldsymbol{k}'). \tag{8}$$

When the volume of the flow region becomes large, we may replace the Fourier sum in (6) by a Fourier integral

$$\sum_{k'} U(\boldsymbol{k}, \boldsymbol{k}') = \int G(k, k')\, \mathrm{d}k' \tag{9}$$

where $G(k, k')$ is the net enstrophy gain by modes of wavenumber k from all modes in the range k' to $k'+\mathrm{d}k'$. In order to write an expression for this quantity, it is necessary to make some assumption about the nonlinear inertial transfer of enstrophy across the spectrum. We use a generalised Von Karman-Heisenberg-von Weizsacker-type model, according to which the process of transfer of enstrophy from large to small wavelengths is described by a gradient-diffusion type cascade process (i.e. a small-scale rapidly adjusting motion superimposed on a large-scale slowly adjusting motion) characterised by an eddy viscosity produced by large wavenumber modes acting to remove enstrophy from small wavenumber modes. This idea is similar to the one originally proposed by Heisenberg (1948) and von Weizsacker (1948) for the transfer of turbulent kinetic energy in the three dimensional case and generalised by Von Karman (1948). The latter model has been shown by Uberoi and Narain (1974) to compare more favourably with experiments than the original Heisenberg-von Weizsacker model.

If each mode in the range of wavenumbers from $k'=k$ to $k'=\infty$ is to make a separate and similar contribution to the eddy viscosity $\tilde{\nu}(k)$ which depends on the energy density $E(k')$ and the wavenumber k' only, then by dimensional considerations, we may write

$$G(k, k') = \begin{cases} 2A[\xi(k')]^{(3/2)-n} k'^{(1/2)-m} [D(k)]^n k^m & k' < k \\ -2A[D(k)]^{(3/2)-n} k^{(1/2)-m} [\xi(k')]^k k'^m & k' > k \end{cases} \tag{10}$$

where A is a universal constant and m and n are arbitrary constants.

The rate of loss of enstrophy by modes with wavenumbers less than some value k is given by

$$\int_0^k \frac{\partial D(k'')}{\partial t}\, \mathrm{d}k'' = -2\nu \int_0^k D(k'')k''^2\, \mathrm{d}k'' - 2\tilde{\nu}(k) \int_0^k [D(k'')]^{(3/2)-n} k''^{(1/2)-m}\, \mathrm{d}k'' \tag{11}$$

where

$$\tilde{\nu}(k) \equiv A \int_k^\infty [\xi(k')]^n k'^m\, \mathrm{d}k'. \tag{12}$$

Let us now replace the left hand side in (11) by the total rate of decay of enstrophy, τ. This is valid for values of k such that

$$\int_0^k D(k'')\, \mathrm{d}k'' \gg \int_k^\infty D(k'')\, \mathrm{d}k''.$$

One then obtains from (11):

$$2\nu D(k)k^2 + [2A(\xi(k))^n k^m]\left(-\tau + 2\nu \int_0^k D(k'')k''^2\, \mathrm{d}k''\right)(2\tilde{\nu}(k))^{-1}$$
$$+ 2\tilde{\nu}(k)[D(k)]^{(3/2)-n} k^{(1/2)-m} = 0. \tag{13}$$

A solution of (13) for arbitrary values of m and n has not been obtained. However, it is possible to obtain the asymptotic forms of solution of (13) in the limit of small and large values of the wavenumber k.

Thus, for small wavenumbers, which correspond to $\nu \ll \tilde{\nu}(k)$, we obtain

$$\xi(k) \sim k^{(4n/3)-(11/3)}. \tag{14}$$

Equation (14) agrees with the well known inertial-range result

$$\xi(k) \sim k^{-3} \tag{15}$$

if we choose

$$n = \tfrac{1}{2} \tag{16}$$

which also corresponds to the choice for n one has to make to reduce (10) to a Heisenberg-von Weizsacker-type model. Thus, the present model has only one free parameter m and reduces completely to the Heisenberg-von Weizsacker-type model by taking $m = -3/2$.

On the other hand, for large wavenumbers, which correpond to $\nu \gg \tilde{\nu}(k)$, equation (13) gives the new dissipative branch

$$\xi(k) \sim k^{-(m-4)/(n-1)} \tag{17}$$

or on using (16),

$$\xi(k) \sim k^{2(m-4)}. \tag{18}$$

For a Heisenberg-von Weizsacker-type model, for which $m = -3/2$, an explicit solution of equation (13) can be obtained:

$$\xi(k) = \left(\frac{2\tau}{A}\right)^{2/3} k^{-3}\left(1 + \frac{4\nu^3}{A^2\tau} k^6\right)^{-4/3}. \tag{19}$$

Equation (19) shows that there is a new length scale ζ,

$$\zeta = \left(\frac{\nu^3}{\tau}\right)^{1/6} \tag{20}$$

that characterises the enstrophy cascade, just as the Kolmogorov scale characterises the energy cascade. Let us call ζ the Kraichnan scale. Equation (19) gives for $k \ll (A^2/4)^{1/6}\zeta^{-1}$,

$$\xi(k) \sim A^{-2/3}\tau^{2/3}k^{-3} \tag{21}$$

in agreement with (15). While (19) gives for $k \gg (A^2/4)^{1/6}\zeta^{-1}$,

$$\xi(k) \sim \frac{A^2\tau^2}{\nu^4} k^{-11} \tag{22}$$

in agreement with (18) when one puts $m = -3/2$.

Equation (18) exhibits a more rapid decay of the spectrum for large wavenumbers. The spectrum in this range, according to (18), is in fact an arbitrarily steep power law. Nonetheless, it is possible to give an even more rapidly decaying exponential-type spectrum using a stationary continuous spectral cascading model.

4. Stationary continuous spectral cascading model

A stationary continuous spectral cascading model gives a satisfactory description of the transfer of turbulent enstrophy at large wavenumbers because the later stages in the cascade tend toward a stationary process in the wavenumber space. (This idea is similar to the one proposed by Pao (1965) for transfer of turbulent kinetic energy at large wavenumbers in the three-dimensional case.)

In stationary turbulence, (6) can be written as

$$N(k) \equiv \int G(k, k')\, \mathrm{d}k' = \nu k^2 D(k) \tag{23}$$

$N(k)$ represents the contribution to the inertial transfer of enstrophy to the mode of wavenumber k from all wavenumbers. Then, the enstrophy flux from wavenumbers less than k to wavenumbers greater than k is

$$R(k) = \int_k^\infty N(k)\, \mathrm{d}k \tag{24}$$

or

$$\frac{\mathrm{d}R}{\mathrm{d}k} = -N(k). \tag{25}$$

If we now visualise the transfer of turbulent enstrophy as a cascading process in which the spectral enstrophy is continuously transferred to ever larger wavenumbers, we can write

$$R(k) = D(K)\frac{\mathrm{d}k}{\mathrm{d}t} \tag{26}$$

where $\mathrm{d}k/\mathrm{d}t$ is the spectral cascading rate. Let us now assume that this process depends on τ (the rate at which the turbulent enstrophy is fed to small eddies), on the viscosity ν (in accordance with (22)), and on the wavenumber k (or equivalently, the size of the small eddies). On dimensional grounds, we then have

$$\frac{\mathrm{d}k}{\mathrm{d}t} = D\tau^{1/3}k \tag{27}$$

where D is a positive constant. This reflects the fact that $\mathrm{d}k/\mathrm{d}t > 0$ for the enstrophy cascade.

Using (25)-(27), (24) becomes

$$\frac{\mathrm{d}}{\mathrm{d}k}[D\tau^{1/3}k^3\xi(k)] = -\nu k^4\xi(k) \tag{28}$$

from which we have

$$\xi(k) = Fk^{-3}\exp\left(-\frac{\nu}{2D\tau^{1/3}}k^2\right). \tag{29}$$

For $k \ll (8D^3)^{1/6}\zeta^{-1}$, (29) gives

$$\xi(k) \approx Fk^{-3} \tag{30}$$

in agreement with (15). Equation (29) gives an exponential decay at very large wavenumbers.

5. β-model for the intermittency corrections to the enstrophy cascade

Though the enstrophy cascades toward small scales through nonlinear interactions, the measure of the spatial domain in which such transfers are active decreases as the scale size decreases (Basdevant and Sadourny 1983, Benzi *et al* 1984). Kraichnan (1971) argued that intermittency will not affect the small-scale energy spectrum because the enstrophy-cascade interaction is not local in wavenumber space. This also means that it takes an infinitely long time to initiate a fully developed spectrum in a nearly inviscid flow driven by random forcing at a fixed wavenumber. However, Basdevant *et al* (1981) and Benzi *et al* (1984) have shown that in the absence of any organised large-scale motion, intermittency is able to steepen the energy spectrum by restoring the spectral localness of nonlinear interactions. This intermittency is the result of the formation of spatially organised vortices, found in the numerical simulations of McWilliams (1984b) in decaying situations after long periods of time, and also in some stationary forced situations with a forcing spectrum at high wavenumbers (Basdevant *et al* 1981, Herring and McWilliams 1985).

Consider now a discrete sequence of scales

$$l_n = l_0 p^{-n} \qquad n = 0, 1, 2, \ldots \tag{31}$$

and a discrete sequence of wavenumbers $k_n = l_n^{-1}$. Here p is the constant ratio of the cascade in sizes. The kinetic energy per unit mass in the nth scale is defined by

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\,\mathrm{d}k. \tag{32}$$

Let us assume that we have a statistically stationary turbulence where enstrophy is introduced into the fluid at scales $\sim l_0$ and is then transferred successively to scales $\sim l_1, l_2, \ldots$, until some scale l_d is reached where dissipation is able to compete with nonlinear transfer. We now assume again that at the nth step, only a fraction β^n of the total space has an appreciable excitation.

The enstrophy per unit mass in the nth scale is then given by

$$D_n \sim \frac{\beta^n V_n^2}{l_n^2} \tag{33}$$

where

$$\beta^n = (p^{D-2})^n \sim \left(\frac{l_n}{l_0}\right)^{2-D}. \tag{34}$$

D is the fractal dimension of the region in which dissipation is concentrated. Equation (34) expresses the fact that intermittency now increases with decrease of scale size.

The rate of transfer of enstrophy per unit mass from the nth scale to the $(n+1)$th scale is given by

$$\tau_n \sim \frac{D_n}{t_n} \sim \frac{\beta^n V_n^3}{l_n^3} \tag{35}$$

where t_n is a characteristic time of the nth scale, $t_n = l_n / V_n$. In the enstrophy inertial range, we assume a stationary process in which enstrophy is introduced at scales $\sim l_0$ and removed at scales $\sim l_d$; conservation of enstrophy requires that

$$\tau_n = \bar{\tau} \qquad l_\mathrm{d} \leqslant l_n \leqslant l_0. \tag{36}$$

It is convenient to think of $\bar{\tau}$ also as the mean enstrophy dissipation rate which is what it would be when the eddies are of the order of the Kraichnan length scale ζ.

Equations (34)-(36) then give

$$V_n \sim \bar{\tau}^{1/3} l_n \left(\frac{l_n}{l_0}\right)^{-(2-D)/3} \tag{37}$$

$$t_n \sim \bar{\tau}^{-1/3} \left(\frac{l_n}{l_0}\right)^{(2-D)/3} \tag{38}$$

$$E_n \sim \bar{\tau}^{2/3} l_n^2 \left(\frac{l_n}{l_0}\right)^{(2-D)/3}. \tag{39}$$

Equation (39) leads to the energy spectrum

$$E(k) \sim \bar{\tau}^{2/3} k^{-3} (k l_0)^{-(2-D)/3}. \tag{40}$$

Equation (40) shows that the intermittency corrections to the enstrophy cascade increase the 3 exponent. This is also in agreement with the predictions for large wavenumbers of the generalised spectral law (13) for the enstrophy cascade. Further, observe that according to (40), the enstrophy cascade cannot have a spectrum steeper than $k^{-11/3}$. The latter result has also been deduced directly from the Navier-Stokes equations (Sulem and Frisch 1975, Pouquet 1978). (The $k^{-11/3}$ spectrum was also shown by Gilbert (1988), to correspond to the passive advection of spiral filaments which form around the coherent vortices observed in numerical simulation of decaying two-dimensional turbulence (McWilliams 1984a).) Thus, intermittency by itself is unable to account fully for steeper spectra observed in the numerical experiments.

Let us now discuss the lower bound for the fractal dimension D in the enstrophy cascade. Equating t_n to the viscous dissipation time, we have

$$\bar{\tau}^{-1/3} \left(\frac{l_n}{l_0}\right)^{(2-D)/3} \sim \frac{l_n^2}{\nu} \tag{41a}$$

or

$$l = l_0 R^{-3/(4+D)} \tag{41b}$$

where

$$R \equiv \frac{\bar{\tau}^{1/3} l_0^2}{\nu}.$$

Now, from (34),

$$D = \frac{\log \beta p^2}{\log p} = \frac{\log N}{\log p} \tag{42}$$

where N is the average number of offspring, which can be less than unity, so that D can assume arbitrary negative values. However, according to (41), there is a dynamical reason to require $D > -4$; otherwise, the enstrophy cascade will never be terminated by viscosity.

Let us next discuss further the manner in which the fractal dimension influences the development and termination of the enstrophy cascade. The first curdling stage leads to curds of size $l_0 p^{-1}$ in which enstrophy dissipation is equal to either 0 or τp^{2-D}, and the Kraichnan scale is $\zeta p^{-(2-D)/6}$. In the nth stage, the average dissipation is

$\tau p^{n(2-D)}$, the curd size is $l_0 p^{-n}$ and the Kraichnan scale is $\zeta p^{-n(2-D)/6}$. Thus, both the Kraichnan scale and the curd size decrease with increase in n. However, curdling can continue only until the curd size is bigger than the Kraichnan scale and will stop thereafter. This occurs when

$$\zeta p^{-n(2-D)/6} \sim l_0 p^{-n}$$

or

$$\zeta / l_0 \sim p^{[1-(2-D)/6]n}. \tag{43}$$

Hence, for the enstrophy cascade, the fractal dimension rules not only the manner in which the curdling proceeds but also the point where it stops.

6. Discussion

We have given a generalised Von Karman–Heisenberg–von Weizsacker-type inertial transfer model for the enstrophy cascade in a two-dimensional turbulence. This model gives spectra that are arbitrarily steep power laws for very high wavenumbers so that this model may be able to provide a satisfactory unified framework for describing both the inertial range and the strongly viscous range of the enstrophy cascade, like the case with three-dimensional turbulence. This aspect is not yet conclusive, since the enstrophy cascade has not been obtained in the laboratory. The large wavenumber limit of the enstrophy cascade can also be modelled in a satisfactory way as a stationary continuous spectral cascading process.

The departures from the Batchelor–Kraichnan scaling law can be described also in terms of intermittency corrections through the so-called β-model which are found to be in qualitative agreement with the predictions made by the generalised spectral law in this paper. However, intermittency by itself has been shown to be unable to account fully for the steeper spectra observed at large wavenumbers in numerical experiments. One may generalise the β-model to admit the possibility that the region containing the dissipation is instead a non-homogeneous fractal. Thus, in the spirit of Mandelbrot's (1976) weighted-curdling model, the contraction factors β may be considered as independent random variables (Benzi *et al* 1984) which can take different values in each scale i at the nth step of the cascade. It is to be noted that though a great deal of work has been done to account for the intermittency corrections, no definite theoretical framework toward this goal exists to date. A deductive theory, based directly on the Navier–Stokes equation, is what is really needed. But this has proved elusive as yet.

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